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Closing of a spin gap and ferromagnetism induced by magnetic impurities: finite-size effects and critical exponents

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Abstract. We consider the one-dimensional gas of electrons interacting via an attractive δ-function potential. The attractive potential leads to the formation of Cooper-pair-like bound states with a spin gap. Without destroying the integrability, we introduce a finite concentration of mixed-valence impurities with two magnetic configurations of spin S and $S + 1/2$, which hybridize with the conduction states of the host. We derive the Bethe *ansatz* equations diagonalizing the system for the correlated host with impurities and discuss the ground-state properties as functions of the concentration of impurities and the Kondo coupling. For a sufficiently large density of electrons the spin gap is reduced by the impurities and is closed at a critical concentration which depends on the impurity spin and the coupling strength. A ferromagnetic phase of unpaired spin-polarized itinerant electrons is induced for impurity concentrations larger than the critical one. The critical behaviour of the correlation functions is studied in the gapped and gapless regions using the mesoscopic energy spectrum and conformal field theory.

1. Introduction

Magnetic impurities in superconductors are unfavourable to the formation of Cooper pairs, and the superconducting gap and the transition temperature gradually reduce as the concentration of impurities increases [1, 2]. The gap closes before superconductivity is completely suppressed, which leads to a phenomenon known as gapless superconductivity [2]. As a consequence of the broken time-reversal symmetry, the impurity forms a bound state inside the gap, which also contributes to the closing of the superconducting gap [3].

A possible explanation for the pseudo-gap or spin-gap anomaly observed in underdoped cuprate superconductors is that they are in an intermediate regime between a BCS superconductor and a condensate of preformed bosons. The one-dimensional (1D) gas of electrons with an attractive δ-function potential [4] or the Hubbard model with attractive U [5, 6] exhibit such preformed Cooper pairs without long-range order at all temperatures. For a sufficiently large density of electrons the magnetic impurities weaken the Cooper pairs and, hence, reduce the spin gap (bound-state energy) [7], which eventually closes, as a function of the concentration of impurities. This is in contrast to the suppression of superconductivity in a BCS superconductor by magnetic impurities via the pair-breaking mechanism [1–3]. Within our model, preformed pairs are only broken up if the impurity concentration is larger than a critical one, c_{cr}, at which the spin gap is closed. In this case the magnetic impurities act like an effective magnetic field and yield a phase in which the unpaired electrons have a spontaneous magnetization (ferromagnetism) in the gapless region.

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The exact solution for the 1D electron gas with δ-function interaction was obtained long ago [8] via nested Bethe *ansätze*, and numerous properties are known for both repulsive and attractive interaction U [9]. An impurity embedded into the electron gas usually destroys the integrability. In recent papers we succeeded in constructing 1D integrable correlated electron gas models with magnetic impurity via the quantum inverse scattering method [10–14]. (Other models of isolated impurities in correlated electron hosts have been studied in references [15].) The advantage of the quantum inverse scattering method is that an integrable model and its Bethe *ansatz* solution can be constructed from a consistent set of scattering matrices satisfying the triangular Yang–Baxter relation. This procedure avoids the tedious coordinate Bethe *ansatz* and the *a priori* search of an integrable Hamiltonian. The condition of integrability imposes restrictions on the impurity, since all scattering matrices have to satisfy the triangular Yang–Baxter relation among themselves. In particular, for the Hubbard model with attractive interaction, a mixed-valence hybridization impurity with two magnetic configurations (the undercompensated Kondo effect) was found to be integrable [14]. We will consider the same type of impurity embedded into the electron gas with a parabolic band. Without losing the integrability, this model is then generalized to a finite concentration of impurities.

The rest of the paper is organized as follows. In section 2 we present the main steps that lead to the integrable model and derive the Bethe *ansatz* equations diagonalizing the system. The restrictions imposed on the scattering matrices by the condition of integrability suppress the reflection. As a consequence, the impurity does not form a bound state inside the spin gap, which is an artifact of our impurity model and a non-generic property. In section 3 we discuss the ground-state properties as functions of the concentration of impurities and the coupling strength. The system displays two phases: (i) the spin-gapped phase, which is described in terms of one Fermi sea of hard-core bosons corresponding to the Cooper-pair-like singlet bound states; and (ii) the gapless phase described in terms of two Dirac seas, one for the singlet pairs and the other one representing unpaired spin-polarized electrons. The finite-size corrections to the ground-state energy and the matrix of dressed charges are derived in section 4 for both phases. In section 5 we study the critical behaviour of correlation functions at large distances and long times across the critical point separating the gapless ferromagnetic and the spin-gapped phases. The correlation functions considered are the pair-charge-density Green's function, the Cooper-pair correlation function, and the spin-density and spin-flip response functions. The Aharonov–Bohm persistent-current oscillation pattern is also discussed. The critical behaviour and the oscillations are qualitatively different for the two phases. Conclusions follow in section 6.

2. Bethe *ansatz* equations

2.1. Model and scattering matrices

As the host we consider the 1D electron gas with a parabolic band and the electrons interacting via an attractive contact interaction:

$$\mathcal{H}_{host} = \sum_{\sigma} \int \mathrm{d}x\ \psi_{\sigma}^{\dagger}(x)(-\partial^2/\partial x^2)\psi_{\sigma}(x) - 2U \int \mathrm{d}x \int \mathrm{d}x'\ \delta(x-x')\psi_{\uparrow}^{\dagger}(x)\psi_{\downarrow}^{\dagger}(x')\psi_{\downarrow}(x')\psi_{\uparrow}(x) \tag{1}$$

where $\psi_{\sigma}^{\dagger}(x)$ creates an electron of spin σ at x, the mass has been equated to $1/2$ and U is the interaction strength. If we consider only two electrons at x_1 and x_2 the wave function in the

sectors $x_1 > x_2$ and $x_1 < x_2$ is related by the scattering matrix for two electrons [8, 9]:

$$\hat{X}(k_1 - k_2) = \frac{(k_1 - k_2)\hat{I} - \mathrm{i}U\hat{P}}{(k_1 - k_2) - \mathrm{i}U} \tag{2}$$

where

$$\hat{I} = \delta_{\sigma_1\sigma_1'}\delta_{\sigma_2\sigma_2'} \qquad \text{and} \qquad \hat{P} = \delta_{\sigma_1'\sigma_2}\delta_{\sigma_2'\sigma_1}$$

are the identity and permutation operators, respectively. Here the unprimed (primed) indices refer to states before (after) scattering and the k_i are the wavenumbers of the plane waves. The sets of wavenumbers of the incoming and outgoing particles are identical. We consider an attractive interaction, i.e. $U > 0$.

The impurity is introduced via its matrix describing the scattering with the itinerant electrons $\hat{S}$:

$$\hat{S}^{\sigma\sigma'}_{MM'}(k - k_0) = \delta_{\sigma\sigma'}\delta_{MM'} - (M\sigma|M + \sigma)(M'\sigma'|M' + \sigma')\frac{\mathrm{i}U(2S + 1)}{k - k_0 + \mathrm{i}U(2S + 1)/2}P^{\sigma\sigma'}_{MM'} \tag{3}$$

where

$$P^{\sigma\sigma'}_{MM'} = \delta_{\sigma\sigma'}\delta_{MM'} + \delta_{-\sigma\sigma'}\delta_{M'M+2\sigma}.$$

Here M and M' are the spin projections of the impurity before and after scattering, respectively. The Clebsch–Gordan coefficient $(M\sigma|M + \sigma)$ which is a shorthand notation for

$$(SM; \tfrac{1}{2}\sigma|S\tfrac{1}{2}(S + \tfrac{1}{2})(M + \sigma)) \tag{4}$$

selects the way in which the impurity couples to the itinerant electrons and preserves the SU(2) invariance of the spin space. The impurity of spin S is capable of temporarily absorbing the spin $\frac{1}{2}$ of the conduction electrons to form an effective spin $(S + \frac{1}{2})$, i.e. it exists in two different spin configurations. Two Clebsch–Gordan coefficients are needed, the first one to absorb the electron and the second one to release the electron again. This is characteristic of intermediate-valence systems, where actual states are the linear superpositions of two electronic configurations [16–18]. In equation (2) the parameter k_0 is the impurity rapidity which controls the degree of 'valence admixture'. Note that equations (2) and (3) are both unitary.

The necessary and sufficient condition for the integrability is the factorization of the many-particle scattering matrix into two-particle scattering matrices, i.e. the scattering matrices satisfy the triangular Yang–Baxter relation [10]:

$$\begin{aligned} X^{\sigma_1\sigma_1'}_{\sigma_2\sigma_2'}(k_1 - k_2)Y^{\sigma_1'\sigma_1''}_{MM'}(k_1 - k_3)Y^{\sigma_2'\sigma_2''}_{M'M''}(k_2 - k_3) \\ = Y^{\sigma_2\sigma_2'}_{MM'}(k_2 - k_3)Y^{\sigma_1\sigma_1'}_{M'M''}(k_1 - k_3)X^{\sigma_1'\sigma_1''}_{\sigma_2'\sigma_2''}(k_1 - k_2) \end{aligned} \tag{5}$$

where the sum over repeated indices is implicit. Here $\hat{Y}$ is either the matrix $\hat{X}$ for scattering of itinerant electrons or the matrix $\hat{S}$ for scattering of a conduction electron and the impurity. If $\hat{Y} \equiv \hat{X}$ the spin components M, M' and M'' are identical to σ_3, σ_3' and σ_3''. On the other hand, if $\hat{Y} \equiv \hat{S}$ the rapidity k_3 is just k_0. The tedious but straightforward verification that the scattering matrices (2) and (3) satisfy the relations (5) has been explicitly carried out in references [16–18].

Consider now N_e itinerant electrons and N_i impurities in a box of length L with periodic boundary conditions. Periodic boundary conditions imposed on a given electron means that it has to interchange position with all other electrons and all the impurities to arrive again at its original position. Each shifting through (permutation) involves a two-particle scattering matrix, such that when the particle is back at the original position we have obtained an operator

that consists of a product of (N_e-1) electron–electron scattering matrices, $\hat{X}$, and N_i impurity scattering matrices, $\hat{S}$, i.e.

$$\hat{T}_j(k_j)=\hat{Y}^{-1}_{j,j+1}(k_j-k_{j+1})\cdots\hat{Y}^{-1}_{j,N}(k_j-k_N)\hat{Y}^{-1}_{j,1}(k_j-k_1)\cdots\hat{Y}^{-1}_{j,j-1}(k_j-k_{j-1}) \tag{6}$$

where $\hat{Y}$ denotes either an electron–electron or an impurity scattering matrix. Here k_j is the wavenumber of the jth electron, and $k_l\equiv k_0$ if $\hat{Y}_{j,l}$ is an electron–impurity scattering matrix. There are N_e such transfer matrices, one for each electron, which have to be diagonalized simultaneously. The periodic boundary conditions for each electron determine the eigenvalue of $\hat{T}_j(k_j)$ to be $\exp(\mathrm{i}k_jL)$.

2.2. Diagonalization

The monodromy matrix [10,18,19] for N_e itinerant electrons and N_i impurities, $N=N_e+N_i$, is defined as

$$L^{\{\sigma_1'\cdots\sigma_N'\}\tau'}_{\{\sigma_1\cdots\sigma_N\}\tau}(\alpha;\alpha_1,\ldots,\alpha_N)=Y^{\tau'\mu_1}_{\sigma_1'\sigma_1}(\alpha_1-\alpha)Y^{\mu_1\mu_2}_{\sigma_2'\sigma_2}(\alpha_2-\alpha)\cdots Y^{\mu_{N-1}\tau}_{\sigma_N'\sigma_N}(\alpha_N-\alpha) \tag{7}$$

with the implicit summation over all of the μ_j-indices. Here $\hat{Y}$ denotes a scattering matrix and α is the spectral parameter. Equation (7) consists of a product of N_e scattering matrices of the $\hat{X}$ type and N_i electron–impurity scattering matrices, which can be arranged in any arbitrary order. The indices σ_i generically denote electron or impurity spins. With respect to the indices τ and τ' the monodromy matrix forms a 2×2 matrix, which we write as $\hat{L}^{\tau'}_{\tau}(\alpha)$ omitting the spin indices and the parameters α_j.

From the Yang–Baxter relations it follows that the monodromy matrix satisfies the identity [10, 18, 19]

$$X^{\tau_1\tau_1'}_{\tau_2\tau_2'}(\alpha-\alpha')\hat{L}^{\tau_1'}_{\tau_3}(\alpha')\hat{L}^{\tau_2'}_{\tau_3}(\alpha)=\hat{L}^{\tau_2}_{\tau_2'}(\alpha)\hat{L}^{\tau_1}_{\tau_1'}(\alpha')X^{\tau_1'\tau_3}_{\tau_2'\tau_3'}(\alpha-\alpha') \tag{8}$$

where the sum over repeated indices is implicit. A transfer matrix is defined as $\hat{T}(\alpha)=\sum_\tau\hat{L}^\tau_\tau(\alpha)$. Using equation (8) and the unitarity of $\hat{X}(\alpha)$, it is straightforward to show that transfer matrices at different α-values commute, $[\hat{T}(\alpha),\hat{T}(\alpha')]=0$, which means that they can all be diagonalized simultaneously. The diagonalization yields the discrete Bethe *ansatz* equations discussed below. Note that these commutation relations also hold if not all impurities have the same rapidity k_0, i.e. it holds for any distribution of k_0. With $\alpha=k_j$, $j=1,\ldots,N_e$, and $\alpha_l=k_l$ being either k_0 or an electron rapidity (N_i of the former and (N_e-1) of the latter kind), equations (6) are just the trace over the monodromy matrix, which as shown above can be diagonalized simultaneously for all spectral parameters.

To derive the Bethe *ansatz* equations we follow the standard procedure outlined elsewhere [19]. The four components of the monodromy matrix with respect to the indices τ and τ' are denoted by

$$\hat{L}^1_1=\hat{A}\qquad\hat{L}^1_2=\hat{B}\qquad\hat{L}^2_1=\hat{C}\qquad\hat{L}^2_2=\hat{D} \tag{9}$$

so the diagonalization of $\hat{T}(\alpha)$ corresponds to diagonalizing $\hat{A}(\alpha)+\hat{D}(\alpha)$. The operators $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$ obey commutation relations which are obtained from equation (8) by explicitly using the two-electron scattering matrix, (2). The results are similar to those derived in references [18, 19] and will not be repeated here. We denote with Ω_0 the state of maximum spin (usually called the vacuum state), i.e., the state in which all electron and impurity spins point upward, $M=S$. The $\hat{C}$-operator acts like a 'spin-raising' operator and when applied to

Ω_0 it yields zero. The diagonal operators satisfy

$$\begin{aligned}
&\hat{A}(\alpha)\Omega_0 = \Lambda_A(\alpha)\Omega_0 \qquad \hat{D}(\alpha)\Omega_0 = \Lambda_D(\alpha)\Omega_0 \\
&\Lambda_A(\alpha) = \left[\frac{k_0 - \alpha - \mathrm{i}U(2S+1)/2}{k_0 - \alpha + \mathrm{i}U(2S+1)/2}\right]^{N_i} \\
&\Lambda_D(\alpha) = \left[\frac{k_0 - \alpha + \mathrm{i}U(2S-1)/2}{k_0 - \alpha + \mathrm{i}U(2S+1)/2}\right]^{N_i} \prod_{j=1}^{N_e} \frac{\alpha_j - \alpha}{\alpha_j - \alpha - \mathrm{i}U}.
\end{aligned} \tag{10}$$

On the other hand, $\hat{B}$ has the properties of a 'spin-lowering' operator, so the vector

$$\Omega(\alpha'_1, \ldots, \alpha'_{M^*}) = \prod_{\beta=1}^{M^*} \hat{B}(\alpha'_\beta)\Omega_0 \tag{11}$$

corresponds to M^* flipped spins and has a spin projection equal to $\frac{1}{2}N_e - M^* + N_i S$.

The Bethe *ansatz* equations are the conditions on the set of parameters $\alpha'_1, \ldots, \alpha'_{M^*}$ under which the vector (11) is an eigenvector of $\hat{A}(\alpha) + \hat{D}(\alpha)$. Applying $\hat{A}(\alpha) + \hat{D}(\alpha)$ to $\Omega(\alpha'_1, \ldots, \alpha'_{M^*})$ and commuting $(\hat{A} + \hat{D})$ through all the $\hat{B}$-operators, one obtains two types of term, namely (i) terms that reproduce the vector (11) and (ii) 'unwanted' terms of the form

$$\sum_{\gamma=1}^{M^*} \Lambda_\gamma(\alpha, \{\alpha'_\beta\}) \prod_{\beta=1, \beta\neq\gamma}^{M^*} \hat{B}(\alpha'_\beta)\hat{B}(\alpha)\Omega_0. \tag{12}$$

Hence, the vector (11) is an eigenvector of $(\hat{A} + \hat{D})$ only if $\Lambda_\gamma(\alpha, \{\alpha'_\beta\}) = 0$ for each γ. The actual expression for $\Lambda_\gamma(\alpha, \{\alpha'_\beta\})$ and the eigenvalue of $(\hat{A} + \hat{D})$, i.e. the amplitude multiplying the term of type (i) that reproduces the vector (11), are obtained by making use of the commutation relations of the operators (9). With $\alpha_l = k_l$ for $l = 1, \ldots, N_e$, $\alpha = k_j$, and setting $\alpha'_\beta = \Lambda_\beta - \mathrm{i}U/2$, we obtain the discrete Bethe *ansatz* equations for the electron gas with N_i impurities

$$\left[\frac{k_j - k_0 - \mathrm{i}(2S+1)U/2}{k_j - k_0 + \mathrm{i}(2S+1)U/2}\right]^{N_i} \mathrm{e}^{\mathrm{i}k_j L} = \prod_{\beta=1}^{M^*} \frac{k_j - \Lambda_\beta - \mathrm{i}U/2}{k_j - \Lambda_\beta + \mathrm{i}U/2} \qquad j = 1, \ldots, N_e \tag{13}$$

$$\left[\frac{\Lambda_\alpha - k_0 - \mathrm{i}US}{\Lambda_\alpha - k_0 + \mathrm{i}US}\right]^{N_i} \prod_{j=1}^{N_e} \frac{\Lambda_\alpha - k_j - \mathrm{i}U/2}{\Lambda_\alpha - k_j + \mathrm{i}U/2} = -\prod_{\beta=1}^{M^*} \frac{\Lambda_\alpha - \Lambda_\beta - \mathrm{i}U}{\Lambda_\alpha - \Lambda_\beta + \mathrm{i}U} \qquad \beta = 1, \ldots, M^*. \tag{14}$$

In each equation the first factor on the left-hand side arises from the impurities. The remaining factors correspond to the electron gas host. The energy of the system is given by $E = \sum_{j=1}^{N_e} k_j^2$.

2.3. Impurity Hamiltonian

Each impurity acts like a localized charged particle of spin S. For simplicity we first consider an isolated impurity. The Hamiltonian describing the interaction of the impurity with a lattice electron gas can in principle be obtained as the derivative of the logarithm of the transfer matrix with respect to the spectral parameter α at $\alpha = 0$. In general, the impurity then interacts with the conduction electrons in both partial waves (right and left movers).

The impurity Hamiltonian has a more appealing form if the kinetic energy is linearized in the momentum about the Fermi level. The interaction of the impurity with conduction states then becomes a contact potential, so the impurity only couples to states with even parity (s waves) with respect to the impurity site. States with odd parity affect the impurity only

indirectly via the Luttinger liquid. The impurity corresponds to a mixed-valence impurity with two magnetic configurations of spins S and $S' = S + \frac{1}{2}$, respectively, hybridized via a conduction electron [16–18]:

$$H_{imp} = \epsilon \sum_{M'} |S'M'\rangle\langle S'M'| + V \sum_{\sigma M M'} (SM, \tfrac{1}{2}\sigma | S\tfrac{1}{2} S'M') \times \int \mathrm{d}x\ \delta(x)[c_\sigma^\dagger(x)|SM\rangle\langle S'M'| + \mathrm{H.c.}] \tag{15}$$

where the bra and ket denote the impurity states of the two configurations, and the Clebsch–Gordan coefficient selects the spin components. The completeness condition for the impurity requires

$$\sum_{M'} |S'M'\rangle\langle S'M'| + \sum_{M} |SM\rangle\langle SM| = 1. \tag{16}$$

Here $\epsilon = |k_0|/v$ represents the energy difference between the two configurations relative to the Fermi level, v is the Fermi velocity and $V^2 = (2S+1)U/v$. The impurity is capable of temporarily absorbing the spin of one conduction electron to form an effective spin $S' = S + \frac{1}{2}$, i.e. the wave function is a linear superposition of two different spin configurations [16–18]. Note that for $S = 0$ the Clebsch–Gordan coefficients are equal to 1 and the impurity is the $U \to \infty$ limit of the Anderson model, except that the host is now a Luttinger liquid.

While in a free-electron host the Anderson impurity has two independent parameters, namely ϵ and V, the integrability in the interacting host fixes V, so there is only one free parameter, namely the impurity rapidity k_0. In contrast, for an impurity in a free-electron gas the charge and spin fluctuations occur on different energy scales. $|k_0|$ is inversely proportional to the Kondo exchange coupling.

In view of the triangular relation, the Bethe *ansatz* equations and the energy of the system only depend on U, the number of itinerant electrons N_e, the number of impurities N_i, the impurity rapidity k_0 and the length of the box L, but not on the space order, i.e. the relative distances between the impurities. This leads to a locality structure, i.e. a large degeneracy of systems having the same energy. This property is the consequence of the integrability and is non-generic, but deviations from the integrability are expected to modify the locality property into a quasi-locality structure, similar to the one experimentally observed for heavy-fermion alloys. This locality structure has previously been discussed for other models [20, 21]. Note also that we have not incorporated additional interactions between impurities, so for the present model the impurities are non-interacting.

3. Ground-state properties

3.1. Integral equations

States of the model are given by solutions of the Bethe *ansatz* equations. It follows that in the ground state the Λ-rapidities are all real, while the k-rapidities are either real (unpaired electrons) or coupled to the Λ-rapidities in 2-strings, $k = \Lambda \pm \mathrm{i}U/2$ (paired electrons). The impurities do not introduce a new class of solutions.

The attractive interaction pairs the electrons into Cooper-like singlet states without off-diagonal long-range order even at $T = 0$ [4–6]. The singlet pairs act like hard-core bosons and are characterized by the set of real Λ-rapidities (2-strings of k-rapidities). The singlet pairs introduce a spin gap (binding energy) in the excitation spectrum of unpaired electrons, which are represented by the real rapidities k. They correspond to unbound spin-polarized electrons. The distribution densities for the rapidities and their holes, i.e. $\rho(k)$ and $\rho_h(k)$

for the unbound electrons, and $\sigma'(\Lambda)$ and $\sigma'_h(\Lambda)$ for the pair rapidities, satisfy the following integral equations [4, 9]:

$$\rho_h(k) + \rho(k) + \int_{-Q}^{Q} \mathrm{d}\Lambda\ a_1(k-\Lambda)\sigma'(\Lambda) = \frac{1}{2\pi} + \frac{1}{2}c[a_{2S+1}(k-k_0) + a_{2S+1}(k+k_0)] \tag{17}$$

$$\begin{aligned}\sigma'_h(\Lambda) + \sigma'(\Lambda) + \int_{-Q}^{Q} \mathrm{d}\Lambda'\ a_2(\Lambda-\Lambda')\sigma'(\Lambda') \\ = -\int_{-B}^{B} \mathrm{d}k\ a_1(\Lambda-k)\rho(k) + \frac{1}{\pi} + \frac{1}{2}c[a_{2S+2}(\Lambda-k_0) + a_{2S+2}(\Lambda+k_0)]\end{aligned} \tag{18}$$

where $c = N_i/L$ is the density of impurities and

$$a_n(x) = (Un/2\pi)/(x^2 + (Un/2)^2)].$$

Equations (17) and (18) follow from the discrete Bethe *ansatz* equations. The driving terms proportional to c arise from the impurities, while the remaining ones are due to the electron gas. The sign of k_0 introduces a forward or backward chirality. We consider here a system without net chirality, i.e. with as many impurities with backward as with forward chirality. This is not a unique choice, but any distribution of signs of k_0 gives qualitatively the same answer. Note that the system is integrable for any distribution of k_0.

The total number of electrons and the magnetization are given by

$$\begin{aligned}\frac{N_e}{L} &= n = \int_{-B}^{B} \mathrm{d}k\ \rho(k) + 2\int_{-Q}^{Q} \mathrm{d}\Lambda\ \sigma'(\Lambda) \\ \frac{S_z}{L} &= Sc + \frac{1}{2}\int_{-B}^{B} \mathrm{d}k\ \rho(k).\end{aligned} \tag{19}$$

The number of electrons and the magnetization of the host determine the integration limits Q and B. Q and B monotonically increase with the filling of the respective bands.

Equations (17) and (18) are linear in the densities, and the driving terms of host and impurities are additive, so the contributions to the densities for the host and the impurities can be separated: $\rho(k) = \rho_{host}(k) + c\rho_{imp}(k)$ and $\sigma'(\Lambda) = \sigma'_{host}(\Lambda) + c\sigma'_{imp}(\Lambda)$. It is useful to introduce the ratios of the densities for the isolated impurity and the pure host, obtained by solving equations (17) and (18):

$$R_\rho = \frac{\rho_{imp}(k)}{\rho_{host}(k)} \qquad R_{\sigma'} = \frac{\sigma'_{imp}(\Lambda)}{\sigma'_{host}(\Lambda)} \tag{20}$$

in terms of which the dressed energies are given by [4]

$$\varepsilon(k) = \left(k^2 - \mu - \frac{1}{2}H\right)\left[1 + cR_\rho\right] - \int_{-Q}^{Q} \mathrm{d}\Lambda\ a_1(\Lambda-k)\psi(\Lambda) \tag{21}$$

$$\begin{aligned}\psi(\Lambda) = 2(\Lambda^2 - U^2/4 - \mu)[1 + cR_{\sigma'}] - \int_{-Q}^{Q} \mathrm{d}\Lambda'\ a_2(\Lambda-\Lambda')\psi(\Lambda') \\ - \int_{-B}^{B} \mathrm{d}k\ a_1(\Lambda-k)\varepsilon(k).\end{aligned} \tag{22}$$

The dressed energies enter the Fermi function, i.e. states with negative (positive) energy are occupied (empty). The Fermi points are given by the zeros of the dressed energies, $\varepsilon(\pm B) = 0$ and $\psi(\pm Q) = 0$, and determine in this way the magnetic field and the chemical potential for given number of electrons and magnetization. The integrations in equations (17), (18), (21) and (22) are over occupied states, i.e. the intervals in which ε and ψ are negative.

The spin gap is half the smallest energy required to depair a singlet bound state and in zero magnetic field it is

$$\text{Gap} = -\mu\left[1 + cR_\rho(0)\right] - \int_{-Q}^{Q} \mathrm{d}\Lambda\ a_1(\Lambda)\psi(\Lambda) \tag{23}$$

where ψ is the solution of equation (22) for $B = 0$.

3.2. Properties

The impurities manifest charge fluctuations characteristic of intermediate valence [14]. This is in part a consequence of the correlations in the host, which drive the valence of the impurities. The valence is maximum if the impurity rapidity k_0 lies in the continuum of the pair-charge rapidities (in resonance with the Fermi sea states), and decreases monotonically with increasing U.

Since the impurities contribute with a fraction of an electron (intermediate valence), the total number of electrons changes with the impurity concentration c if the integration limits B and Q are kept constant. Hence, the distributions of rapidities for the host and the impurity do not change as a function of c. Similarly, one could consider the system at a fixed number of electrons and readjust Q and B with c. Numerically, we found this effect to be small within the range of concentrations used here.

The spin fluctuations are suppressed by the spin gap of the host. There is no response to a magnetic field smaller than the critical field H_c (corresponding to the depairing energy of Cooper-singlet bound states or the gap) and the magnetization of the impurities equals S. For fields slightly larger than H_c, the magnetic susceptibility of the impurity has a square-root divergence [14], revealing the van Hove singularity of the empty unpaired electron band of the host. This behaviour differs drastically from the ordinary Kondo effect.

Usually a magnetic impurity introduces a bound state inside the gap of a BCS superconductor. The spin gap represents (half of) the energy gained by forming a Cooper bound state, although without long-range order. The impurities considered here are only elastic scatterers (as required by the integrability), and hence they do not form a bound state. An impurity that includes both elastic scattering and reflection would give rise to a bound state in the spin gap. This aspect of our impurity is non-generic, imposed by the condition of integrability. We expect, however, some other properties of the impurity, e.g. the lack of spin screening (a consequence of the spin gap) and the pair weakening (because they are preformed pairs), to be valid more generally.

Consider first equations (21) and (22) in the limits $Q \to 0$ and $B \to 0$. Then $\mu = -U^2/4$ and the gap is given by $(U^2/4)(1+cR_\rho(0))$. Hence, for a very low density of electrons the gap increases with c. The magnitude of the gap decreases with the density of electrons, $n = N_e/L$, indicating that the Cooper pairs become more weakly bound as n increases. The rate of change of the gap with the concentration of impurities also changes with n as shown in figure 1, and becomes negative for sufficiently large n. Since we are interested in the closing of the gap, we will consider this latter situation.

The integral equations (17), (18), (21) and (22) were solved numerically for a fixed number of host electrons ($n = 0.68$) and zero magnetic field. The dressed energies are shown in figure 2 as functions of the rapidities for three impurity concentrations, $k_0 = 1.0$ and $S = 0$. This corresponds to Anderson impurities of a spin $S' = 1/2$. The dressed energy for the Cooper pairs ψ increases with c in the relevant regime of occupied states, but does not display any qualitative changes. The dressed energy ε, representing the unpaired spin-polarized electrons, on the other hand, decreases with c in the neighbourhood of $k \approx 0$. For $c = c_{cr}$, the critical

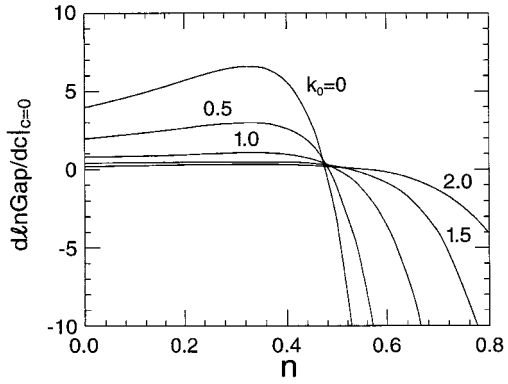

Figure 1. Rate of change of the spin gap with the concentration of impurities as a function of the density of electrons for several values of the impurity rapidity k_0, $S' = 1/2$ and $U = 1$. Note that for small n the gap increases with c as a consequence of the renormalization of the density of states, while for large n there is pair weakening.

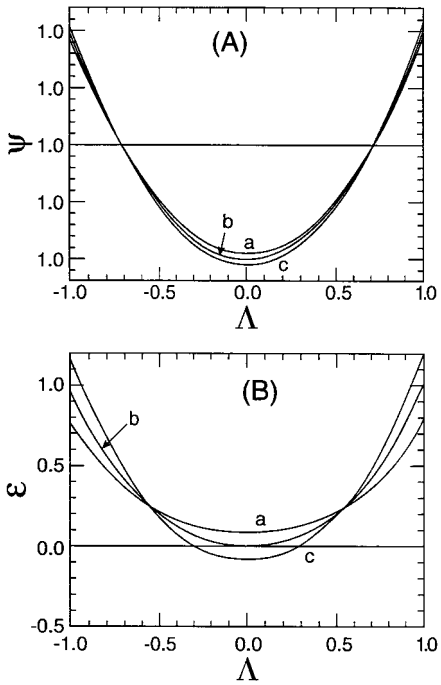

Figure 2. (A) Dressed energy ψ for the singlet pairs as a function of the rapidity Λ and (B) the dressed energy ε for the unpaired electrons as a function of k for $U = 1$, $S = 0$ ($S' = 1/2$), $k_0 = 1.0$ and $Q = 0.716$. Three impurity concentrations are shown: (a) $c = 0$, (b) $c = c_{cr} = 0.1$, and (c) $c = 0.2$. Note that ε is negative for $c > c_{cr}$ and small k.

concentration, we have $\varepsilon(0) = 0$. Hence, for $c < c_{cr}$ the energy is always positive and the states are unoccupied. In this case the value of $\varepsilon(0)$ represents the gap, i.e. the energy required to place one additional electron into the unpaired electron band. In order to break up a pair an energy of at least $2\varepsilon(0)$ is needed. For $c > c_{cr}$ the unpaired electron band is partially filled

and gives rise to a spontaneous magnetization of the itinerant electrons (ferromagnetism). In this sense the effect of the impurities is similar to that of an external magnetic field.

The magnitude of the spin gap and the spontaneous magnetization as a function of impurity concentration are shown in figure 3(A) for $S = 0$, $U = 1$ and impurity rapidity $k_0 = 1.0$. The spin gap decreases linearly with c. If the equations are solved for a constant number of electrons (here fixed at $n = 0.68$ per site), there are small deviations from linearity in c, requiring an adjustment of Q for each concentration. The spin gap closes at c_{cr}, the critical concentration. The spin gap is reduced via pair weakening, rather than by pair breaking as for magnetic impurities in a BCS superconductor. No unpaired electrons are generated as long as there is a spin gap. The ground-state impurity magnetization for $c < c_{cr}$ is $N_i S$. For impurity concentrations larger than c_{cr} a fraction of the itinerant electrons are depaired and spontaneously magnetized. For c slightly above c_{cr} the total magnetization of the electron gas with impurities is $S_z/L = cS + \alpha(c - c_{cr})^{1/2}$, where α is a constant (see figure 3(A)).

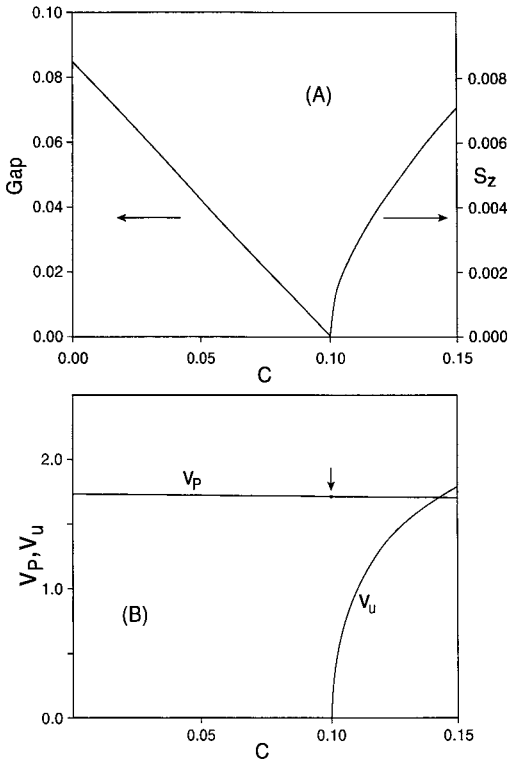

Figure 3. (A) Spin gap and spontaneous magnetization of the unpaired itinerant electrons (per site) and (B) group velocities, v_p for the Cooper pairs and v_u for the unpaired electrons, as functions of the impurity concentration. The model parameters are $U = 1$, $S = 0$ ($S' = 1/2$), $Q = 0.716$ and $k_0 = 1.0$. The spin gap closes at c_{cr}, while S_z and v_u are only non-trivial for $c > c_{cr}$.

The group velocities of the two classes of particles are given by

$$v_p = \frac{(\partial\psi/\partial\Lambda)_Q}{2\pi\sigma'(Q)} \qquad v_u = \frac{(\partial\varepsilon/\partial k)_B}{2\pi\rho(B)} \tag{24}$$

and are displayed in figure 3(B) as functions of c. Note that v_u is not defined for $c < c_{cr}$ and increases proportional to $\sqrt{c - c_{cr}}$ for c slightly above c_{cr}, and v_p has a small discontinuity in its slope at c_{cr}.

4. Finite-size corrections

The finite size of a ring manifests itself in several ways:

(i) Impurities are important in mesoscopic systems, since their contribution to extensive quantities can become large and observable, and may even change the properties.
(ii) The finite length of a ring yields persistent currents with oscillation periods given by interference patterns of the Aharonov–Bohm type.
(iii) Finite-size corrections to the energy determine the critical exponents of the asymptotic dependence at long times and large distances of correlation functions via conformal field theory [22–26].

To calculate the finite-size corrections to the ground-state energy, i.e. the change of the energy due to the finite length of the ring, we follow the procedure developed in references [22] and [27]. The ground-state energy including mesoscopic terms is given by

$$E = L\epsilon_\infty + \sum_l \frac{\pi v_l}{2L}\left\{\left[\sum_q (\hat{z}^{-1})_{ql}\Delta N_q\right]^2 - \frac{1}{3}\right\} + \sum_l \frac{2\pi v_l}{L}\left\{\left[\sum_q z_{lq}(D_q + \vartheta_q)\right]^2 + n_l^+ + n_l^-\right\} \tag{25}$$

where ϵ_∞ is the ground-state energy density in the thermodynamic limit, l and q label the two bands and take values p and u, and v_l denote the group velocities of the two bands. The quantities ϑ_q are phase shifts of the Aharonov–Bohm type, induced by the magnetic field flux through the ring.

The mesoscopic energy depends on several quantum numbers. ΔN_q is the departure of the number of particles in the band q from the equilibrium value. D_q is the backward-scattering quantum number, i.e. $2D_q$ represents the difference of forward- and backward-moving states from the equilibrium value (backward scattering). These quantities are sensitive to the parity in each set of rapidities. Finally, $n_q^\pm$ define the low-lying particle–hole excitations about each of the Fermi points. Here ΔN_q, $n_q^\pm$ and $2D_q$ always take integer values; hence D_q can either be an integer or half-integer depending on the initial conditions.

The quantities z_{lq} in equation (25) are the dressed generalized charges of the excitations at the Fermi points, $z_{uq} = \xi_{u,q}(B)$ and $z_{pq} = \xi_{p,q}(Q)$. The dressed charges are determined by the integral equations [27]

$$\begin{aligned}&\xi_{u,q}(k) + \int_{-Q}^{Q} \mathrm{d}\Lambda\, a_1(\Lambda - k)\xi_{p,q}(\Lambda) = \delta_{u,q}\\ &\xi_{p,q}(\Lambda) + \int_{-Q}^{Q} \mathrm{d}\Lambda'\, a_2(\Lambda - \Lambda')\xi_{p,q}(\Lambda') + \int_{-B}^{B} \mathrm{d}k\, a_1(\Lambda - k)\xi_{u,q}(k) = \delta_{p,q}.\end{aligned} \tag{26}$$

In equation (25) $\hat{z}^{-1}$ denotes the inverse of the matrix of dressed generalized charges. The dressed charges describe the interplay of the different Fermi points when particles are added or removed.

The integral equations (26) depend on c, S and k_0 only through the integration limits B and Q. We solved equations (26) numerically for $S' = 1/2$ ($S = 0$), zero magnetic field, $n = 0.5$, $U = 1$ and $k_0 = 0.5$. For $c < c_{cr}$ only the band of Cooper pairs is filled and, hence, only z_{pp} is defined. The dressed generalized charges are shown in figure 4(A) as functions of c. From (26) it is clear that z_{uu} and z_{pp} are always positive, while z_{pu} and z_{up} are negative. $z_{pp}(c)$ has a discontinuity in its slope at c_{cr}, z_{pu} vanishes at c_{cr} and $z_{uu} = 1$ at this point.

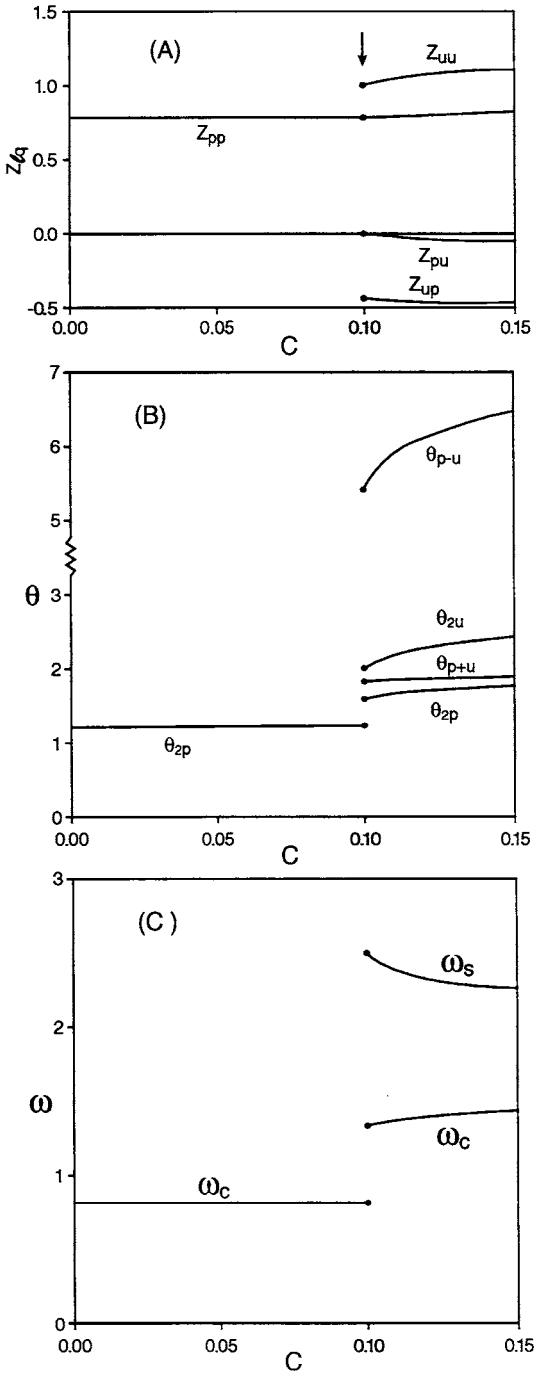

Figure 4. (A) Dressed generalized charges, (B) critical exponents for the pair-charge-density correlation function and (C) leading critical exponents of the Cooper-pair correlation function, ω_c, and the spin-flip response function, ω_s, as functions of c. The model parameters are $U = 1$, $S = 0$ ($S' = 1/2$), $Q = 0.716$ and $k_0 = 1.0$. Only z_{pp} is defined for $c < c_{cr}$. The (non-integer) exponents are discontinuous across the transition.

In terms of the quantum numbers defined above, the total momentum of the system is given by

$$P = \frac{2\pi}{L}\sum_l \left[N_l(D_l + \vartheta_l) + n_l^+ - n_l^-\right]. \tag{27}$$

Equations (25) and (27) define the mesoscopic corrections to the ground-state energy and momentum of a two-component Luttinger liquid.

5. Correlation functions

We now calculate first the long-time large-distance asymptotics of the pair-charge-density and Cooper-pair correlation functions, then we obtain the spin-density and spin-flip response functions, and finally we briefly discuss the Aharonov–Bohm oscillation pattern.

For gapless one-dimensional systems (Luttinger liquids) the correlations fall off as power laws of time and distance. The critical exponents follow from conformal field theory in combination with the finite-size excitation spectrum. The conformal dimensions of a primary field $\varphi_{\tilde{Q}}$, characterized by a set of quantum numbers $\tilde{Q}$, are obtained from equations (25) and (27):

$$2\Delta_l^{\pm} = \left[\frac{1}{2}\sum_q (\hat{z}^{-1})_{ql}\Delta N_q \pm \sum_q z_{lq} D_q\right]^2 + 2n_l^{\pm}. \tag{28}$$

From an inspection of the discrete Bethe *ansatz* equations it follows that $2D_u = \Delta N_p - \Delta N_u$ (mod 1) and $2D_p = \Delta N_u$ (mod 1). The asymptotic form of a correlation function for the operator $\varphi(x,t)$ is [23–26]

$$\langle\varphi(x,t)\varphi(0,0)\rangle = \sum_{\tilde{Q}} B(\tilde{Q}) \exp\left[-2\mathrm{i}x\sum_l p_l D_l\right]\prod_l (x - \mathrm{i}v_l t)^{-2\Delta_l^+}(x + \mathrm{i}v_l t)^{-2\Delta_l^-} \tag{29}$$

where the index $\tilde{Q}$ refers to the conformal fields contained in the operator φ, and $p_u = \pi n_u$ and $p_p = \pi n_p$ are the Fermi momenta of the two bands. Here n_l is the number of particles in the band l.

The pair-density operator, defined by the Cooper-pair number operator $n_p(x,t)$, does not change the number of particles in either band, i.e. $\Delta N_u = \Delta N_p = 0$, and hence D_u and D_p are integers. We have to consider the pair-density correlation function in the two regimes, $c < c_{cr}$ and $c > c_{cr}$. For $c < c_{cr}$ only the band of pairs is populated, so only the conformal dimensions

$$2\Delta_p^{\pm} = 2n_p^{\pm} + z_{pp}^2 D_p^2$$

play a role, and the leading terms of the pair-density correlation function are

$$\langle n_p(x,t)n_p(0,0)\rangle = n_p^2 + A\frac{x^2 - v_p^2t^2}{(x^2 + v_p^2t^2)^2} + B_2\frac{\cos(2p_px + \varphi_2)}{(x^2 + v_p^2t^2)^{z_{pp}^2}} + B_4\frac{\cos(4p_px + \varphi_4)}{(x^2 + v_p^2t^2)^{4z_{pp}^2}} + \cdots \tag{30}$$

where n_p is the expectation value of the density operator, φ_2 and φ_4 are two phases, and A, B_2 and B_4 are amplitudes. The term with amplitude A corresponds to $D_p = 0$, $n_p^+ = 1$, $n_p^- = 0$, and $n_p^+ = 0$, $n_p^- = 1$. The other two terms are oscillatory and arise from $D_p = 1$ and 2, respectively, with $n_p^+ = n_p^- = 0$, and represent a backward scattering of 2 and 4 times the Fermi momentum.

The situation changes drastically for $c > c_{cr}$, where the two Dirac seas interfere. The conformal dimensions for the pair density are now

$$2\Delta_u^{\pm} = 2n_u^{\pm} + (z_{uu}D_u + z_{up}D_p)^2 \qquad 2\Delta_p^{\pm} = 2n_p^{\pm} + (z_{pu}D_u + z_{pp}D_p)^2$$

and the leading asymptotic terms of the correlation function are

$$\begin{aligned}\langle n_p(x,t)n_p(0,0)\rangle = {} & n_p^2 + A_p\frac{x^2 - v_p^2t^2}{(x^2+v_p^2t^2)^2} + A_u\frac{x^2 - v_u^2t^2}{(x^2+v_u^2t^2)^2} \\ & + B_{2p}\frac{\cos(2p_px+\varphi_{2p})}{(x^2+v_p^2t^2)^{z_{pp}^2}(x^2+v_u^2t^2)^{z_{up}^2}} + B_{2u}\frac{\cos(2p_ux+\varphi_{2u})}{(x^2+v_p^2t^2)^{z_{pu}^2}(x^2+v_u^2t^2)^{z_{uu}^2}} \\ & + B_{4p}\frac{\cos(4p_px+\varphi_{4p})}{(x^2+v_p^2t^2)^{4z_{pp}^2}(x^2+v_u^2t^2)^{4z_{up}^2}} + B_{4u}\frac{\cos(4p_ux+\varphi_{4u})}{(x^2+v_p^2t^2)^{4z_{pu}^2}(x^2+v_u^2t^2)^{4z_{uu}^2}} \\ & + C_+\frac{\cos[2(p_p+p_u)x+\varphi_+]}{(x^2+v_p^2t^2)^{(z_{pp}+z_{pu})^2}(x^2+v_u^2t^2)^{(z_{uu}+z_{up})^2}} \\ & + C_-\frac{\cos[2(p_p-p_u)x+\varphi_-]}{(x^2+v_p^2t^2)^{(z_{pp}-z_{pu})^2}(x^2+v_u^2t^2)^{(z_{uu}-z_{up})^2}} + \cdots . \end{aligned} \tag{31}$$

Charge-density waves now propagate with two different group velocities, v_p and v_u, and oscillating terms with 2 and 4 times the Fermi momentum of each band, as well as sums and differences of the Fermi momenta, appear. The amplitudes A_u, B_{2u} and B_{4u} are expected to be small for the pair-density correlation function. Note that the charge density for unpaired electrons has the same quantum numbers as the pair density, so for $c > c_{cr}$ the general form is again (31).

Of special interest is the equal-time correlation function. For $t = 0$ and suppressing the phases φ we have for $c < c_{cr}$

$$\langle n_p(x)n_p(0)\rangle = n_p^2 + \frac{A}{x^2} + B_2\frac{\cos(2p_px)}{x^{\theta_{2p}}} + \cdots \tag{32}$$

and for $c > c_{cr}$

$$\begin{aligned}\langle n_p(x)n_p(0)\rangle = {} & n_p^2 + \frac{A}{x^2} + B_{2p}\frac{\cos(2p_px)}{x^{\theta_{2p}}} + B_{2u}\frac{\cos(2p_ux)}{x^{\theta_{2u}}} \\ & + C_+\frac{\cos[2(p_p+p_u)x]}{x^{\theta_{p+u}}} + C_-\frac{\cos[2(p_p-p_u)x]}{x^{\theta_{p-u}}} + \cdots . \end{aligned} \tag{33}$$

The critical exponents θ are shown in figure 4(B) as functions of c. Note that the dominant term in all cases oscillates with $2p_p$, i.e. involves simple backward scattering across the Fermi surface of the Cooper hard-core bosons.

The operator $O^\dagger(x,t) = \psi_\uparrow^\dagger(x,t)\psi_\downarrow^\dagger(x,t)$ creates a Cooper pair at x and time t. It involves $\Delta N_p = 1$ and $\Delta N_u = 0$, and hence D_p is integer, while D_u is a half-integer. For $c < c_{cr}$ we obtain for the Cooper-pair correlation function

$$\langle O^\dagger(x,t)O(0,0)\rangle = \frac{A}{(x^2+v_p^2t^2)^{1/4z_{pp}^2}} + \sum_{\alpha=\pm} B_\alpha \mathrm{e}^{-2\mathrm{i}\alpha p_px}\prod_{\beta=\pm}[x - \mathrm{i}\beta v_pt]^{-(z_{pp}-\alpha\beta/2z_{pp})^2} + \cdots \tag{34}$$

where the three terms correspond to D_p equal to 0, +1 and −1, respectively. The leading term is the one of amplitude A. For $c > c_{cr}$, on the other hand, the two Fermi seas interfere and we have ($D_p = 0$ and $D_u = \pm\frac{1}{2}$)

$$\begin{aligned}\langle O^\dagger(x,t)O(0,0)\rangle = {} & \sum_{\alpha=\pm} A_\alpha \mathrm{e}^{-\mathrm{i}\alpha p_ux}\prod_{\beta=\pm}[x - \mathrm{i}\beta v_pt]^{-\frac{1}{4}(z_{pu}+\alpha\beta z_{uu}/\det)^2} \\ & \times [x - \mathrm{i}\beta v_ut]^{-\frac{1}{4}(z_{uu}-\alpha\beta z_{pu}/\det)^2} + \cdots \end{aligned} \tag{35}$$

where 'det' is the determinant of the matrix of dressed charges. In the limit of an empty unpaired electron band these two terms reduce to the first term in (34). Note that the addition or removal of a Cooper pair shakes up the Dirac sea of unpaired electrons and long-wavelength oscillations (p_u is small) arise.

Again, the equal-time correlation function has a simpler form, i.e. $\langle O^\dagger(x)O(0)\rangle = A/x^{\omega_c} + \cdots$ for $c < c_{cr}$ and $\langle O^\dagger(x)O(0)\rangle = A\cos(p_u x)/x^{\omega_c} + \cdots$ for $c > c_{cr}$. The leading critical exponent ω_c is displayed in figure 4(C) as a function of c. Note that the non-integer exponents for all of the correlation functions are discontinuous across the transition. Typically the correlations fall off faster with distance in the ferromagnetic two-Fermi-sea regime.

The spin-density operator is $\frac{1}{2}n_u$. Its quantum numbers are $\Delta N_p = \Delta N_u = 0$, i.e. the same ones as for the pair-density operator. The terms in principle allowed by symmetry in the asymptote of the correlation function have then the same form as in equations (30) and (31). However, for $c < c_{cr}$ there is no spin density at $T = 0$ and the correlation function is identically zero. For $c > c_{cr}$, on the other hand, the general form is (31) with $\frac{1}{2}n_u$ replacing n_p. Now the amplitudes A_p, B_{2p} and B_{4p} are expected to be small, but the exponents are the same.

The spin-flip operator $S^+(x,t) = \psi_\uparrow^\dagger(x,t)\psi_\downarrow(x,t)$ destroys one Cooper pair ($\Delta N_p = -1$) and creates two unpaired electrons ($\Delta N_u = 2$). The backward-scattering quantum number D_p is an integer and D_u is a half-integer. The leading asymptotic terms correspond to $D_p = n_p^\pm = n_u^\pm = 0$ and $D_u = \pm\frac{1}{2}$, i.e.

$$\langle S^+(x,t)S^-(0,0)\rangle = \sum_{\alpha=\pm} A_\alpha \mathrm{e}^{-\mathrm{i}\alpha p_u x} \prod_{\beta=\pm} [x - \mathrm{i}\beta v_p t]^{-\frac{1}{4}(z_{pu}-\alpha\beta(z_{uu}+2z_{up})/\det)^2} \times [x - \mathrm{i}\beta v_u t]^{-\frac{1}{4}(z_{uu}+\alpha\beta(z_{pu}+2z_{pp})/\det)^2} + \cdots. \tag{36}$$

The function has a long-wavelength oscillation. The equal-time correlation function is

$$\langle S^+(x)S^-(0)\rangle = A\cos(p_u x)/x^{\omega_s}$$

and the critical exponent ω_s is displayed in figure 4(C) as a function of c.

Finally, we consider persistent currents of the Aharonov–Bohm type, which are induced in response to a magnetic field flux ϕ threading the ring. The flux introduces the phase shifts ϑ_q in equation (25). In the absence of excitations the mesoscopic correction to the ground-state energy is given by [28]

$$\Delta E = \sum_l \frac{2\pi v_l}{N_a}\left[\left(z_{lu}\left\{\frac{\phi}{\phi_0}\right\} + z_{lp}\left\{\frac{2\phi}{\phi_0}\right\}\right)^2 - \frac{1}{12}\right] \tag{37}$$

where the curly brackets confine the phase to the interval $|\varphi| < 0.5$ (modulo an integer). Here $\phi_0 = hc/e$ is the elemental magnetic flux quantum and Cooper pairs acquire a phase of $2\phi/\phi_0$ because they involve two electron charges. The persistent current is defined as $j_c = -\partial\,\Delta E/\partial\phi$.

For $c < c_{cr}$ all electrons are paired in the ground state and

$$\Delta E = (2\pi v_p/N_a)[z_{pp}^2\{2\phi/\phi_0\}^2 - 1/12].$$

As a function of ϕ the mesoscopic energy is continuous and piecewise parabolic with a periodicity of $\phi_0/2$, and consequently the persistent current has a simple sawtooth pattern. The charge stiffness or Drude weight is $D = 2v_p z_{pp}^2/\pi$.

For $c > c_{cr}$, on the other hand, there are two Fermi seas coupling to the flux, so there are two periods of oscillation, namely ϕ_0 and $\phi_0/2$. The energy is no longer continuous and jumps at $\phi_0/4$ and $3\phi_0/4$ (and is periodic), due to the interplay of the two Fermi surfaces. The discontinuities of the energy are mesoscopic, of the order of the Heisenberg uncertainty, and

have been discussed previously in a different context, namely for the Heisenberg chain with impurities [29]. The persistent current is the superposition of two oscillations of the sawtooth form. The persistent current has δ-function-like singularities where the energy jumps. These singularities can be interpreted as supercurrents, necessary to generate the discontinuities in the energy. The discontinuities disappear at finite T or with impurity scattering, since this suppresses the higher-order-harmonic content in the oscillations.

6. Conclusions

We have studied the critical behaviour of correlation functions at large distances and long times across the critical point separating the spin-gapped and the gapless ferromagnetic phases of the interacting 1D electron gas with a finite concentration of magnetic impurities. In the ground state the system has two phases, namely a spin-gapped phase consisting of one Fermi sea of hard-core bosons (Cooper-pair-like singlet bound states) and a gapless phase with two Dirac seas, one corresponding to the singlet pairs and the other one representing unpaired spin-polarized electrons.

We calculated the mesoscopic corrections to the ground-state energy for a finite system and obtained the critical exponents for the pair-charge-density Green's function, the Cooper-pair correlation function, and the spin-density and spin-flip response functions. The pair-density and Cooper-pair correlation functions are non-trivial in both phases. The critical exponents are discontinuous at the transition. In the ferromagnetic phase the Cooper-pair response function has a long-wavelength modulation that is proportional to the magnetization. The ground-state spin correlation functions are zero in the gapped phase, but fall off with a power law in the ferromagnetic phase.

The Aharonov–Bohm persistent-current oscillation pattern has also been studied. The persistent current is a simple sawtooth in the gapped phase, while in the gapless phase the interference of the two Fermi surfaces gives rise to a superposition of two sawtooth patterns with different periodicities and mesoscopic discontinuities in the energy.

Our impurity model has restrictions imposed by the integrability, which requires pure elastic scattering without reflection. As a consequence the impurity does not give rise to a bound state inside the spin gap, which is an artifact of our impurity model and non-generic. Furthermore, we have not considered the possibility of weak localization due to disorder [30]. In the electron gas with attractive U, the impurities effectively act similarly to a magnetic field, closing the gap and polarizing the depaired electrons. In our model we have not included an explicit interaction between the impurities.

On the other hand, a finite concentration of magnetic impurities introduced into a correlated host without a spin gap, e.g. the supersymmetric t–J model, are antiferromagnetically correlated as experimentally found in some heavy-fermion compounds [20]. In this case the spins can rearrange (absence of a gap) and form a magnetic singlet ground state. We have also studied the possibility of antiferromagnetic correlations among the impurity spins in the system with a spin gap [31]. For such a situation to occur, additional interactions, compatible with the Bethe *ansatz* equations, have to be introduced.

Acknowledgments

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